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On the stability and instability of linear potential systems subjected to infinitesimal circulatory forces

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The stability of linear multi-degree-of-freedom stable potential systems with multiple natural frequencies under the action of infinitesimal circulatory forces is considered. Contrary to the received view that such systems are inherently unstable, a careful study shows that such systems have a much more complex behaviour than previously recognized and could exhibit an alternation of stability and instability that depends on the structure of the potential system and its interaction with the circulatory forces. The conditions under which stability or instability ensues and the nature of this alternation in stability are explicitly obtained. In low-dimensional stable potential systems, when the coefficients of the circulatory forces are proportional to an arbitrarily small scalar parameter, all the circulatory forces that cause flutter instability are described.

1. Introduction

The area of stability of dynamical systems is at the crossroads of physics, mathematics and engineering. The physicist is interested in instabilities that arise in nature, the mathematician's interest lies in exact mathematical formulations that provide conditions under which instabilities occur, and the engineer is interested in the design of engineered systems that are stable. The stability of linear potential systems in the presence of positional circulatory forces, in a general setting, was first brought to attention by the

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outstanding work of Merkin in the 1950s in which he demonstrated that when all the vibrational frequencies of such a system are identical, instability ensues [1]. This foundational result in the theory of stability was followed by the observation that when the potential matrix (that describes the potential forces) and the circulatory matrix (that describes the circulatory forces) commute then instability follows [2]. More recently, it was discovered that such commutation can occur if and only if the potential matrix has at least one multiple frequency of vibration and that just one such multiple frequency of vibration of a potential system is sufficient to make it unstable under infinitesimal circulatory perturbative matrices that commute with the potential matrix [3]. Such a situation of having multiple vibrational frequencies can, and often does, arise in complex multi-degree-of-freedom (MDOF) systems such as spacecraft and building structures in which, say, the fourth bending frequency coincides with the second torsional frequency of vibration of the structure.

Though the subject has been investigated for several decades, our knowledge of the conditions for stability and instability of linear potential systems subjected to positional perturbative forces is apparently still far from complete. This is highlighted by the recent observation that such instabilities can be induced even when the potential and perturbatory matrices do not commute and are not necessarily circulatory [4]. Routes to instability caused by such positional forces, both finite and infinitesimal, are shown to be dependent on the interaction of the positional perturbative matrices (forces) and the potential matrix through the potential matrix's eigen structure. The removal of the necessity of delving into the eigen structure of potential matrices has only recently been accomplished for MDOF systems subjected to circulatory forces. An instability criterion that uses instead the 'gross' property of the rank of the products of potential and perturbatory matrices has been obtained, thereby providing a further generalization of Merkin's classical result [5].

The present paper considers a general linear MDOF potential system that has only a single multiple frequency of vibration—something not too uncommon in complex large-scale systems such as spacecraft, and building structures subjected to strong earthquake ground shaking—and considers the effect of general infinitesimal positional circulatory perturbations on it. As opposed to several studies that deal with low-dimensional systems that have typically two to four degrees of freedom, we consider a general MDOF potential system and explore its stability under circulatory forces.

It has been generally believed hereto that such systems, albeit simple since we do not consider other types of forces like damping, are 'all' unstable [6,7]. The analysis presented herein, however, shows that their stability/instability has far greater complexity than has been understood to date. It is shown that depending on the nature and structure of the potential system, it could remain stable or become unstable, and explicit conditions when such stability/instability occurs are obtained. Most vibratory systems are, of course, acted upon by additional forces, such as, damping and gyroscopic forces. Our lack of basic understanding of MDOF potential systems in the absence of these additional forces, i.e. when subjected only to infinitesimal circulatory positional forces, has prompted us, as a first step, to not include their effects in this sequel. Much like the classical foundational result first obtained by Merkin, which is now one of the cornerstones of stability theory (see Krechtnikov & Marsden [8]) the results obtained herein are thus seen to be, in a sense, fundamental, providing a sort of a baseline from which future studies on the stability of MDOF potential systems that include other types of additional forces can be addressed.

We begin by considering the potential system described by the equation

$$\tilde{M}\ddot{q} + \tilde{K}q = 0, \tag{1.1}$$

where the n by n matrix $\tilde{M}$ is a positive definite matrix and $\tilde{K}$ is a real symmetric matrix. The n-vector of generalized coordinates is denoted by q, and the dots indicate differentiation. Potential systems are of considerable importance in physics and engineering because most real-life systems, those that occur naturally and those that are engineered, are usually modelled as potential systems. The addition of a perturbing circulatory force to such a system results in the system described by the equation

$$\tilde{M}\ddot{q}+\tilde{K}q+\varepsilon\tilde{N}q=0, \tag{1.2}$$

where $\tilde{N}$ is a real constant skew-symmetric matrix and ε is a dimensionless parameter which is introduced to characterize the intensity of the circulatory force described by $-\tilde{N}q$. Making the transformation $x=\tilde{M}^{1/2}q$, where the exponent ½ indicates the unique positive definite square root of the matrix $\tilde{M}$, and premultiplying equations (1.1) and (1.2) by $\tilde{M}^{-1/2}$, we get the following equations that describe the potential system and the perturbed potential system

$$\ddot{x}+Kx=0 \tag{1.3}$$

and

$$\ddot{x}+Kx+\varepsilon Nx=0, \tag{1.4}$$

where the symmetric matrix $K=\tilde{M}^{-1/2}\tilde{K}\tilde{M}^{-1/2}$ and skew-symmetric $N=\tilde{M}^{-1/2}\tilde{N}\tilde{M}^{-1/2}$. Clearly, system (1.2) is equivalent to system (1.4), and we shall from here on consider this system.

It is well-known that the potential system is stable, i.e. every solution $x(t)$ of equation (1.3) is bounded for all non-negative t, if and only if the potential matrix K is positive definite ($K>0$), and in what follows, we will assume that this condition is satisfied. On the other hand, the following is known about the stability of system (1.4).

(a) The system (1.4) is unstable by flutter, i.e. there exists oscillating motion with a growing amplitude, if $\varepsilon^2||N||_F^2>||K||_F^2-1/n(\mathrm{Trace}K)^2$, where $||.||_F$ denotes the Frobenius norm [9]. This says that the introduction of *sufficiently large* circulatory forces into a stable potential system always destroys its stability.

(b) If

$$|\varepsilon|||N||_2<\frac{1}{2}\min_{1\le i\ne j\le n}|\lambda_i-\lambda_j|, \tag{1.5}$$

where $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of the potential matrix K and $||.||_2$ denotes the spectral norm, then the system (1.4) is stable [10]. This means that a stable potential system *all* of whose natural frequencies are distinct remains stable after the addition of *sufficiently small* circulatory forces (also, see [11]).

In the case of multiple natural frequencies of a potential system, the following has recently been formulated.

(c) Let the potential matrix K have a single eigenvalue λ_0 with multiplicity $m\ge 2$, and let $T=[T_p|T_r]$ be an orthogonal matrix, where the $n\times p$ submatrix T_p contains any $2\le p\le m$ eigenvectors of K corresponding to the multiple eigenvalue, and the $n\times r$ submatrix T_r contains the remainder (i.e. $r=n-p$) of the eigenvectors of K. Then, if the following conditions hold

$$T_p^TNT_p\ne 0,\quad T_p^TNT_r=0, \tag{1.6}$$

the system (1.4) is unstable by flutter [4]. This result contains, as a special case, the famous Merkin's theorem [1], which assumes the commutativity of the matrices K and N [2,3,12]. It should be noted that in applications, the condition given in (1.6) relies on an analysis of the eigen structure of the matrix K, and an appropriate choice of the orthonormal eigenvectors that are to be included in the submatrix T_p [5]. Recently, a rank condition that is equivalent to (1.6) but obviates the need to examine the eigen structure and to pick the proper orthonormal vectors in T_p has been developed [5].

It should be noted that under conditions (1.6) the instability follows for every $\varepsilon\ne 0$, i.e. for circulatory forces of arbitrary intensities, including, of course, intensities that are infinitesimally small. The following simple example shows that a system (1.4) with multiple natural frequencies that do not satisfy the conditions of the above assertion can remain stable when the circulatory forces are sufficiently small.

Example 1.1. Let

$$K = \mathrm{diag}(1, 1, 3) \quad \text{and} \quad N = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}. \tag{1.7}$$

For this example, clearly, $p = m = 2$ and

$$T_p^T N T_p = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad T_p^T N T_r = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and, consequently, conditions (1.6) are not satisfied. On the other hand, it is easy to see that the characteristic equation of this system has the following roots:

$$\pm i, \quad \pm i\sqrt{2 \pm \sqrt{1 - 2\varepsilon^2}},$$

which all are purely imaginary and distinct if $1 - 2\varepsilon^2 > 0$. Hence the systems (1.4) and (1.7) is stable if $|\varepsilon| < \sqrt{2}/2$.

The following question, which first was clearly formulated by Udwadia [3], arises: when do *arbitrarily small* circulatory forces cause flutter instability in stable linear potential systems that have multiple natural frequencies?

This question is not only of importance in itself, but has important ramifications in real-life engineering systems [4]. A *partial* answer to this question is given by criterion (c), which, we emphasize again, includes circulatory forces of arbitrary intensities, not only intensities that are small and/or infinitesimal. The following example clearly shows that the class of infinitesimal circulatory forces causing instability is wider than that proposed by the aforementioned criterion.

Example 1.2. Let

$$K = \mathrm{diag}(1, 1, 4) \quad \text{and} \quad N = \begin{pmatrix} 0 & 1 & 6 \\ -1 & 0 & 0 \\ -6 & 0 & 0 \end{pmatrix}. \tag{1.8}$$

For this example, clearly, $p = m = 2$ and

$$T_p^T N T_p = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad T_p^T N T_r = \begin{bmatrix} 6 \\ 0 \end{bmatrix},$$

and, consequently, the second condition of (1.6) is not satisfied. On the other hand, it can be shown, for example, using the criterion in [13], that system (1.4), (1.8) is unstable and exhibits flutter if $|\varepsilon| \in (0, a) \cup (b, \infty)$, and stable if $|\varepsilon| \in (a, b)$, where $a = 0.157\ldots$ and $b = 0.253\ldots$.

The precise formulation of the stability problem that this article addresses can then be stated as follows.

Problem. In system (1.4), let the positive definite potential matrix K have one multiple eigenvalue of multiplicity $m \geq 2$. Under what conditions is system (1.4) unstable (stable) for arbitrarily small non-zero $|\varepsilon|$?

One way of approaching this problem is by studying the stability for systems of the form $\ddot{x} + Px = 0$, where the matrix P is non-symmetric and depends on parameters, using bifurcation analysis, based on the first approximation of perturbed eigenvalues [7,14]. According to this point of view, points in the parameter space at which the system has multiple imaginary eigenvalues correspond to singularities of the stability boundary. It is shown that for a double semi-simple imaginary eigenvalue the region of flutter instability lies inside a cone with the apex at the singular point [14]. More recently, it has been shown that when the parameters move along the stability boundary the second approximation must also be considered [15].

The approach developed in this paper is based on classical perturbation theory. We assume that a small parameter ε separates the pure circulatory (skew-symmetric) matrix from the potential matrix and take into account, when necessary, higher order approximations (up to order 5). The approach places no limits on the multiplicity of the natural frequencies. It is shown in §2 that this

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perturbational approach provides an answer to the question posed. In §3, we use these results for a dynamical system with less than 5 degrees of freedom ($n < 5$) and obtain all skew-symmetric matrices N that cause flutter instability in potential systems with multiple eigenvalues in which the circulatory forces are described as in (1.4) by εNq where $|\varepsilon|$ is arbitrarily small.

2. Main results

We begin with a few statements that are relevant to our later considerations.

Lemma 2.1. *For some value of the real parameter ε, the system (1.4) is stable if and only if all eigenvalues of the matrix $K+\varepsilon N$ are positive and simple or semi-simple (i.e. the number of linearly independent eigenvectors associated with a multiple eigenvalue of the matrix $K+\varepsilon N$ coincides with its algebraic multiplicity).*

Proof. See, for example, [14]. ■

Remark 2.2. Since $K>0$, the system cannot be unstable due to divergence, i.e. if the system is unstable, then it is flutter unstable with an exponential or polynomial growing amplitude.

Lemma 2.3. *Let $\lambda_i>0$ be eigenvalues of the potential matrix K. Suppose that $\lambda_1=\lambda_2=\cdots=\lambda_m=\lambda_0$, $m\geq 2$, and that the other eigenvalues $\lambda_{m+1},\ \dots\ ,\lambda_n$ are simple. Then, in some neighbourhood of $\varepsilon=0$, the matrix $K+\varepsilon N$ has m eigenvalues $\mu_j(\varepsilon)$ such that $\mu_j(0)=\lambda_0$, $j=1,\ \dots\ ,m$, and $n-m$ simple real and positive eigenvalues $\mu_{m+k}(\varepsilon)$ such that $\mu_{m+k}(0)=\lambda_{m+k}$, $k=1,\ \dots\ ,n-m$.*

Proof. It follows from the Bauer–Fike localization theorems (see, for example [10, Lemma 1]). ■

Therefore for small enough $|\varepsilon|$, the nature of m eigenvalues $\mu_j(\varepsilon)$ of the matrix $K+\varepsilon N$ resulting from the 'splitting' of the multiple eigenvalue λ_0 ($\mu_j(0)=\lambda_0, j=1,\ \dots\ ,m$) determines the character of the stability of the system under consideration. More precisely, a direct consequence of Lemma 2.1 and Lemma 2.3 is the following assertion.

Lemma 2.4. *Suppose that the positive definite potential matrix K has one eigenvalue λ_0 of multiplicity $m\geq 2$, and that its other eigenvalues are simple. If all eigenvalues $\mu_j(\varepsilon)$ of the matrix $K+\varepsilon N$, such that $\mu_j(0)=\lambda_0$, are real and simple or semi-simple in some neighbourhood of $\varepsilon=0$, then the system remains stable for small enough $|\varepsilon|$; otherwise it will be unstable by flutter for arbitrarily small non-zero $|\varepsilon|$.*

We suppose, as in the above, that the potential matrix K has one eigenvalue λ_0 of multiplicity $m\geq 2$, and that the other eigenvalues are simple. Let $T=[T_m|T_{n-m}]$ be an orthogonal matrix, where the $n\times m$ submatrix T_m contains m eigenvectors of K corresponding to the eigenvalue λ_0, and the $n\times(n-m)$ submatrix T_{n-m} contains the remainder $n-m$ of the eigenvectors of K corresponding to the eigenvalues $\lambda_i\neq\lambda_0, i=m+1,\dots,n$. The orthogonal matrix T reduces K and N to the forms

$$\hat{\Lambda}=T^TKT=\mathrm{diag}(\lambda_0 I_m,\Lambda_{n-m}),\quad \hat{N}=T^TNT=\begin{bmatrix}T_m^T\\ T_{n-m}^T\end{bmatrix}[N][T_m|T_{n-m},]:=\begin{bmatrix}\hat{N}_{11} & \hat{N}_{12}\\ -\hat{N}_{12}^T & \hat{N}_{22}\end{bmatrix},\qquad (2.1)$$

where the $(n-m)$-dimensional diagonal matrix $\Lambda_{n-m}=T_{n-m}^TKT_{n-m}$ contains all the eigenvalues of K that are *distinct* from λ_0, and $\hat{N}_{11}=T_m^TNT_m=-\hat{N}_{11}^T$, $\hat{N}_{12}=T_m^TNT_{n-m}$, $\hat{N}_{22}=T_{n-m}^TNT_{n-m}=-\hat{N}_{22}^T$. Let $\mu(\varepsilon)$ be an eigenvalue of the matrix $(\hat{\Lambda}+\varepsilon\hat{N})$ for which $\mu(0)=\lambda_0$ and let $w(\varepsilon)$ be corresponding eigenvector, i.e.

$$(\hat{\Lambda}+\varepsilon\hat{N})w=\mu w.\qquad (2.2)$$

For the Taylor series of $\mu(\varepsilon)$ and $w(\varepsilon)$ about $\varepsilon=0$, we write

$$\mu(\varepsilon)=\lambda_0+\varepsilon\lambda^{(1)}+\varepsilon^2\lambda^{(2)}+\varepsilon^3\lambda^{(3)}+\cdots\qquad (2.3)$$

and

$$w(\varepsilon)=w^{(0)}+\varepsilon w^{(1)}+\varepsilon^2 w^{(2)}+\cdots.\qquad (2.4)$$

Substituting (2.3) and (2.4) into (2.2) and collecting coefficients of equal powers of ε, we find

$$\{\varepsilon^0, \varepsilon^1\} \quad (\hat{\Lambda} - \lambda_0 I)w^{(0)} = 0, (\hat{\Lambda} - \lambda_0 I)w^{(1)} = (\lambda^{(1)} I - \hat{N})w^{(0)}, \tag{2.5}$$

$$\{\varepsilon^2\} \quad (\hat{\Lambda} - \lambda_0 I)w^{(2)} = (\lambda^{(1)} I - \hat{N})w^{(1)} + \lambda^{(2)} w^{(0)}, \tag{2.6}$$

$$\{\varepsilon^3\} \quad (\hat{\Lambda} - \lambda_0 I)w^{(3)} = (\lambda^{(1)} I - \hat{N})w^{(2)} + \lambda^{(2)} w^{(1)} + \lambda^{(3)} w^{(0)}, \tag{2.7}$$

$$\{\varepsilon^4\} \quad (\hat{\Lambda} - \lambda_0 I)w^{(4)} = (\lambda^{(1)} I - \hat{N})w^{(3)} + \lambda^{(2)} w^{(2)} + \lambda^{(3)} w^{(1)} + \lambda^{(4)} w^{(0)} \tag{2.8}$$

and

$$\{\varepsilon^5\} \quad (\hat{\Lambda} - \lambda_0 I)w^{(5)} = (\lambda^{(1)} I - \hat{N})w^{(4)} + \lambda^{(2)} w^{(3)} + \lambda^{(3)} w^{(2)} + \lambda^{(4)} w^{(1)} + \lambda^{(5)} w^{(0)}, \tag{2.9}$$

or more succinctly as

$$\{\varepsilon^k\} \quad (\hat{\Lambda} - \lambda_0 I)w^{(k)} = (\lambda^{(1)} I - \hat{N})w^{(k-1)} + \sum_{j=2}^{k} \lambda^{(j)} w^{(k-j)} \quad k = 2, 3, \ldots. \tag{2.10}$$

We now denote $w^{(j)} = [\,\bar{w}^{(j)T}\ \tilde{w}^{(j)T}\,]^T$, where $\bar{w}^{(j)}$ and $\tilde{w}^{(j)}$ are m and $(n-m)$ dimensional vectors, respectively, and we consider the pair of equations given in (2.5), which can be rewritten as

$$\begin{bmatrix} 0 & 0 \\ 0 & \Lambda_{n-m} - \lambda_0 I_{n-m} \end{bmatrix} \begin{bmatrix} \bar{w}^{(0)} \\ \tilde{w}^{(0)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.11}$$

and

$$\begin{bmatrix} 0 & 0 \\ 0 & \Lambda_{n-m} - \lambda_0 I \end{bmatrix} \begin{bmatrix} \bar{w}^{(1)} \\ \tilde{w}^{(1)} \end{bmatrix} = \begin{bmatrix} \lambda^{(1)} I - \hat{N}_{11} & -\hat{N}_{12} \\ \hat{N}_{12}^T & \lambda^{(1)} I - \hat{N}_{22} \end{bmatrix} \begin{bmatrix} \bar{w}^{(0)} \\ \tilde{w}^{(0)} \end{bmatrix}. \tag{2.12}$$

Then, from the last $(n-m)$ equations in (2.11), we get $(\Lambda_{n-m} - \lambda_0 I_{n-m})\tilde{w}^{(0)} = 0$ from which it follows that $\tilde{w}^{(0)} = 0$. Also, from the first m and the last $(n-m)$ equations in (2.5), we get, respectively, $(\hat{N}_{11} - \lambda^{(1)} I_m)\bar{w}^{(0)} = 0$, and $(\Lambda_{n-m} - \lambda_0 I_{n-m})\tilde{w}^{(1)} = \hat{N}_{12}^T \bar{w}^{(0)}$ since $\tilde{w}^{(0)} = 0$. Thus (2.11) and (2.12) yield the relations

$$\tilde{w}^{(0)} = 0, \tilde{w}^{(1)} = D\hat{N}_{12}^T \bar{w}^{(0)}, \quad \text{and} \quad (\hat{N}_{11} - \lambda^{(1)} I_m)\bar{w}^{(0)} = 0, \tag{2.13}$$

where the diagonal matrix $D = (\Lambda_{n-m} - \lambda_0 I_{n-m})^{-1}$.

It follows from the last relation in (2.13) that $\lambda^{(1)}$ is an eigenvalue of the m by m skew-symmetric matrix $\hat{N}_{11} = T_m^T N T_m$. If $\hat{N}_{11} \neq 0$, then it has at least one pair of imaginary eigenvalues of the form $\pm i\nu$, with $\nu > 0$, and, in view of (2.3), in some neighbourhood of $\varepsilon = 0$, the matrix $\hat{\Lambda} + \varepsilon\hat{N}$ (and, of course, $K + \varepsilon N$) has at least one complex conjugate pair of eigenvalues of the form

$$\mu(\varepsilon) = \lambda_0 \pm i\varepsilon\nu + o(\varepsilon). \tag{2.14}$$

This, taking into account lemma 2.4, leads to our first result.

Result 2.5. *The addition of a circulatory matrix N to the n degree of freedom potential system described by equation (1.3) that has an eigenvalue* λ_0 *of multiplicity m with* $2 \leq m \leq n$ *will cause the system described by equation (1.4) to become unstable by flutter for arbitrarily small non-zero values of* $|\varepsilon|$ *if*

$$\hat{N}_{11} = T_m^T N T_m \neq 0, \tag{2.15}$$

where the columns of the n by m matrix T_m *are orthonormal eigenvectors of K corresponding to the multiple eigenvalue* λ_0. *That is, the work done by the circulatory force under displacements in the subspace spanned by the columns of* T_m *is non-zero. The order of the eigenvectors in the matrix* T_m *is arbitrary.*

It is interesting to note that the introduction of a circulatory force to the potential system can add energy to it and can make it unstable. Indeed, the rate at which the circulatory force $-\varepsilon N x$ is doing work is given by $\varepsilon x^T N \dot{x}$. Along a mode $e^{\lambda t} w$ of the system (a solution of equation (1.4)), where $\lambda = \alpha + i\beta$ and $w = X + iY$ with $X, Y \in \Re^n$, this quantity becomes $\varepsilon\beta e^{2\alpha t} Y^T N X$. Taking for λ a square root of $-\mu$, where μ is determined by equation (2.14), the last expression reduces to the form $\varepsilon\nu\sqrt{\lambda_0} e^{2\alpha t}(1 + O(\varepsilon))$, where $i\nu$ is a non-zero eigenvalue of the skew-symmetric matrix

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in equation (2.15) and $\alpha = O(\varepsilon)$. This confirms that under conditions of result 2.5, the system performs an unbounded oscillatory motion along which the energy in the system grows.

Remark 2.6. In the case $m = 2$, this result may also be established by an application of the results given in [14, Chapter 4], concerning the singularities on the stability boundary of a multi-parameter circulatory system (see also [6]). Also, we note that the expansion (2.14) can be obtained by an application of a classical result for perturbation of a semi-simple multiple eigenvalue of an arbitrary matrix [16, Theorem 2.5], [17, Section 11.7, Theorem 1]; see also [7, Theorem 2.7].

Remark 2.7. It is easy to generalize result 2.5 to the case when the matrix K has several multiple eigenvalues. Indeed, let $\lambda_1, \ldots, \lambda_k$be eigenvalues of K with multiplicities $m_1, \ldots, m_k$, respectively, $m_j \geq 1$, $m_1 + \cdots + m_k = n$. Let $T = [T_{m_1}|T_{m_2}|\ldots|T_{m_k}]$ be an orthogonal matrix, where the $n \times m_j$ submatrix T_{m_j} contains m_j eigenvectors of K corresponding to the eigenvalue λ_j of multiplicity m_j. Then, the system described by equation (1.4) is unstable by flutter for arbitrarily small non-zero values of $|\varepsilon|$ if at least one of the following matrices

$$T_{m_j}^T N T_{m_j}, \quad j = 1, \ldots, k,$$

is non-zero.

In what follows, we consider the case when

$$\hat{N}_{11} = T_m^T N T_m = 0. \tag{2.16}$$

Then, it follows from (2.13) that $\lambda^{(1)} = 0$ because $\bar{w}^{(0)} \neq 0$. On the other hand, putting (2.16) and $\lambda^{(1)} = 0$ in (2.6) and taking into account of (2.13), we get

$$\begin{bmatrix} 0 & 0 \\ 0 & \Lambda_{n-m} - \lambda_0 I_{n-m} \end{bmatrix} \begin{bmatrix} \bar{w}^{(2)} \\ \tilde{w}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & -\hat{N}_{12} \\ \hat{N}_{12}^T & -\hat{N}_{22} \end{bmatrix} \begin{bmatrix} \bar{w}^{(1)} \\ \tilde{w}^{(1)} \end{bmatrix} + \lambda^{(2)} \begin{bmatrix} \bar{w}^{(0)} \\ 0 \end{bmatrix}, \tag{2.17}$$

whose first m equations give $\hat{N}_{12}\tilde{w}^{(1)} = \hat{N}_{12} D \hat{N}_{12}^T \bar{w}^{(0)} = \lambda^{(2)} \bar{w}^{(0)}$. The last $(n - m)$ equations in (2.17) give $\tilde{w}^{(2)} = D\hat{N}_{12}^T \bar{w}^{(1)} - D\hat{N}_{22}\tilde{w}^{(1)}$, so that we get the two relations

$$(S - \lambda^{(2)} I_m)\bar{w}^{(0)} = 0 \quad \text{and} \quad \tilde{w}^{(2)} = D\hat{N}_{12}^T \bar{w}^{(1)} - D\hat{N}_{22}\tilde{w}^{(1)}, \tag{2.18}$$

in which the m by m matrix

$$S = S^T = \hat{N}_{12} D \hat{N}_{12}^T. \tag{2.19}$$

It follows from (2.18) that $\lambda^{(2)}$ is an eigenvalue of the symmetric matrix S.

If the matrix S has all distinct eigenvalues $\lambda_i^{(2)}$, $i = 1, \ldots, m$, then in some neighbourhood of $\varepsilon = 0$, the matrix $\hat{\Lambda} + \varepsilon\hat{N}$ has m real distinct eigenvalues of the forms

$$\mu_i(\varepsilon) = \lambda_0 + \varepsilon^2 \lambda_i^{(2)} + o(\varepsilon^2), \quad i = 1, \ldots, m. \tag{2.20}$$

Thus, in view of Lemma 2.4, we have the following result.

Result 2.8. *If $\hat{N}_{11} = 0$ and all eigenvalues of the symmetric matrix S in* (2.19) *are distinct, then the system described by equation* (1.4) *is stable for arbitrarily small values of $|\varepsilon|$.*

Lemma 2.9. *When $\hat{N}_{11} = 0$ we can generalize* (2.18) *to*

$$(S - \lambda^{(2)} I_m)\bar{w}^{(r-2)} = \hat{N}_{12} D \hat{N}_{22} \tilde{w}^{(r-2)} - \hat{N}_{12} D \sum_{j=2}^{r-1} \lambda^{(j)} \tilde{w}^{(r-1-j)} + \sum_{j=3}^{r} \lambda^{(j)} \bar{w}^{(r-j)}, \quad r = 3, 4, \ldots \tag{2.21}$$

and

$$\tilde{w}^{(r)} = D\hat{N}_{12}^T \bar{w}^{(r-1)} - D\hat{N}_{22}\tilde{w}^{(r-1)} + D \sum_{j=2}^{r} \lambda^{(j)} \tilde{w}^{(r-j)}, \quad r = 2, 3, \ldots. \tag{2.22}$$

Proof. Equation (2.10) with $k=r>2$ gives

$$\begin{bmatrix} 0 & 0 \\ 0 & \Lambda_{n-m}-\lambda_0 I_{n-m} \end{bmatrix}\begin{bmatrix} \bar{w}^{(r)} \\ \tilde{w}^{(r)} \end{bmatrix}=\begin{bmatrix} 0 & -\hat{N}_{12} \\ \hat{N}_{12}^T & -\hat{N}_{22} \end{bmatrix}\begin{bmatrix} \bar{w}^{(r-1)} \\ \tilde{w}^{(r-1)} \end{bmatrix}+\sum_{j=2}^{r}\lambda^{(j)}\begin{bmatrix} \bar{w}^{(r-j)} \\ \tilde{w}^{(r-j)} \end{bmatrix}, \tag{2.23}$$

whose last $(n-m)$ equations and first m equations yield, respectively, result in (2.22) and in

$$-\hat{N}_{12}\tilde{w}^{(r-1)}+\sum_{j=2}^{r}\lambda^{(j)}\bar{w}^{(r-j)}=0. \tag{2.24}$$

But from (2.22), by setting $r\to r-1$, we get

$$\tilde{w}^{(r-1)}=D\hat{N}_{12}^T\bar{w}^{(r-2)}-D\hat{N}_{22}\tilde{w}^{(r-2)}+D\sum_{j=2}^{r-1}\lambda^{(j)}\tilde{w}^{(r-1-j)}, r=3,4,\ldots. \tag{2.25}$$

Substituting this expression in (2.24) and using (2.19) gives

$$-\hat{N}_{12}D\hat{N}_{12}^T\bar{w}^{(r-2)}+\lambda^{(2)}\bar{w}^{(r-2)}+\hat{N}_{12}D\hat{N}_{22}\tilde{w}^{(r-2)}-\hat{N}_{12}D\sum_{j=2}^{r-1}\lambda^{(j)}\tilde{w}^{(r-1-j)}+\sum_{j=3}^{r}\lambda^{(j)}\bar{w}^{(r-j)}=0,$$

which yields (2.21). ■

Now suppose that the matrix S has a single multiple eigenvalue s_0 of multiplicity r, $2\le r\le m$. We put $\lambda^{(2)}=s_0$ in (2.18), which becomes

$$(S-s_0 I_m)\bar{w}^{(0)}=0. \tag{2.26}$$

Using (2.21) for $r=3$ and noting (2.13), we get $(S-\lambda^{(2)}I_m)\bar{w}^{(1)}=\hat{N}_{12}D\hat{N}_{22}\tilde{w}^{(1)}+\lambda^{(3)}\bar{w}^{(0)}$, so that

$$(S-s_0 I_m)\bar{w}^{(1)}=(G+\lambda^{(3)}I_m)\bar{w}^{(0)}, \tag{2.27}$$

where the m by m skew-symmetric matrix

$$G=-G^T=\hat{N}_{12}D\hat{N}_{22}D\hat{N}_{12}^T. \tag{2.28}$$

Let $\tilde{T}=[\tilde{T}_r|\tilde{T}_{m-r}]$ be an m by m orthogonal matrix, where the $m\times r$ submatrix $\tilde{T}_r$ contains r eigenvectors of the symmetric m by m matrix S corresponding to the multiple eigenvalue s_0, and the $m\times(m-r)$ submatrix $\tilde{T}_{m-r}$ contains the remainder $m-r$ of the eigenvectors of S. The matrix $\tilde{T}$ transforms matrices (2.19) and (2.28) to the forms

$$\hat{S}=\tilde{T}^T S\tilde{T}=\mathrm{diag}(s_0 I_r,\bar{\Lambda}_{m-r}),\quad \hat{G}=\tilde{T}^T G\tilde{T}=\begin{bmatrix} \hat{G}_{11} & \hat{G}_{12} \\ -\hat{G}_{12}^T & \hat{G}_{22} \end{bmatrix}, \tag{2.29}$$

where $\bar{\Lambda}_{m-r}=\tilde{T}_{m-r}^T S\tilde{T}_{m-r}$ and $\hat{G}_{11}=\tilde{T}_r^T G\tilde{T}_r$, $\hat{G}_{12}=\tilde{T}_r^T G\tilde{T}_{m-r}$, $\hat{G}_{22}=\tilde{T}_{m-r}^T G\tilde{T}_{m-r}$. The skew-symmetric matrix $\hat{G}_{11}$ is r by r. Setting $\bar{w}^{(j)}=\tilde{T}[\bar{u}^{(j)T}\quad \tilde{u}^{(j)T}]^T$, where $\bar{u}^{(j)}$ and $\tilde{u}^{(j)}$ are r and $(m-r)$ dimensional vectors, respectively, in (2.26) and (2.27) and premultiplying these by $\tilde{T}^T$ we get

$$\begin{bmatrix} 0 & 0 \\ 0 & \bar{\Lambda}_{m-r}-s_0 I_{m-r} \end{bmatrix}\begin{bmatrix} \bar{u}^{(0)} \\ \tilde{u}^{(0)} \end{bmatrix}=0, \tag{2.30}$$

and

$$\begin{bmatrix} 0 & 0 \\ 0 & \bar{\Lambda}_{m-r}-s_0 I_{m-r} \end{bmatrix}\begin{bmatrix} \bar{u}^{(1)} \\ \tilde{u}^{(1)} \end{bmatrix}=\begin{bmatrix} \hat{G}_{11}+\lambda^{(3)}I_r & \hat{G}_{12} \\ -\hat{G}_{12}^T & \hat{G}_{22}+\lambda^{(3)}I_{m-r} \end{bmatrix}\begin{bmatrix} \bar{u}^{(0)} \\ \tilde{u}^{(0)} \end{bmatrix}. \tag{2.31}$$

Note that (2.30) and (2.31) have the same 'structure' as (2.11) and (2.12), and in a manner similar to what we got in (2.13), we get

$$\tilde{u}^{(0)}=0,\quad \tilde{u}^{(1)}=-\hat{D}\hat{G}_{12}^T\bar{u}^{(0)},\quad (\hat{G}_{11}+\lambda^{(3)}I_r)\bar{u}^{(0)}=0, \tag{2.32}$$

where $\hat{D}=(\bar{\Lambda}_{m-r}-s_0 I_{m-r})^{-1}$.

It follows from the last equation in (2.32) that r eigenvalues of the r by r skew-symmetric matrix $\hat{G}_{11} = \tilde{T}_r^T G \tilde{T}_r$ are the coefficients $\lambda^{(3)}$ in expansion (2.3) with $\lambda^{(1)} = 0$ and $\lambda^{(2)} = s_0$. If this matrix $\hat{G}_{11}$ is non-zero, then it has at least one pair of conjugate purely imaginary eigenvalues $\pm ig$, with $g > 0$, so that the matrix $K + \varepsilon N$ has at least one complex conjugate pair of eigenvalues of the form

$$\mu(\varepsilon) = \lambda_0 + \varepsilon^2 s_0 \pm i\varepsilon^3 g + o(\varepsilon^3). \tag{2.33}$$

Thus, in view of (2.33), since $g \neq 0$, again according to lemma 2.4, we have the following result.

Result 2.10. *If $\hat{N}_{11} = 0$ and $\hat{G}_{11} = \tilde{T}_r^T G \tilde{T}_r \neq 0$, then the system described by equation (1.4) is unstable by flutter for arbitrarily small non-zero values of $|\varepsilon|$.*

If $\hat{G}_{11} = \tilde{T}_r^T G \tilde{T}_r = 0$, then it follows from (2.32) that $\lambda^{(3)} = 0$. In this case, from (2.21) for $r = 4$, we get

$$(S - \lambda^{(2)} I_m)\bar{w}^{(2)} = \hat{N}_{12} D \hat{N}_{22} \tilde{w}^{(2)} - \lambda^{(2)} \hat{N}_{12} D \tilde{w}^{(1)} + \lambda^{(4)} \bar{w}^{(0)},$$

which upon on setting $\lambda^{(2)} = s_0$ and using (2.13) and (2.18) becomes

$$\begin{aligned}(S - s_0 I_m)\bar{w}^{(2)} &= \underbrace{\hat{N}_{12} D \hat{N}_{22} D \hat{N}_{12}^T}_{G} \bar{w}^{(1)} - \underbrace{(s_0 \hat{N}_{12} D^2 \hat{N}_{12}^T + \hat{N}_{12} D \hat{N}_{22} D \hat{N}_{22} D \hat{N}_{12}^T)}_{R} \bar{w}^{(0)} + \lambda^{(4)} \bar{w}^{(0)} \\ &= G \bar{w}^{(1)} - R \bar{w}^{(0)} + \lambda^{(4)} \bar{w}^{(0)}.\end{aligned} \tag{2.34}$$

Here, we have denoted the m by m symmetric matrix

$$R = s_0 \hat{N}_{12} D^2 \hat{N}_{12}^T + \hat{N}_{12} D \hat{N}_{22} D \hat{N}_{22} D \hat{N}_{12}^T = s_0 \hat{N}_{12} D^2 \hat{N}_{12}^T + \hat{N}_{12} (D \hat{N}_{22})^2 D \hat{N}_{12}^T. \tag{2.35}$$

Writing $\bar{w}^{(j)} = \tilde{T}[\bar{u}^{(j)T} \quad \tilde{u}^{(j)T}]^T$, $j = 1, 2, 3$, premultiplying (2.34) by $\tilde{T}^T$, and noting from (2.32) that $\tilde{u}^{(0)} = 0$, we get

$$\begin{bmatrix} 0 & 0 \\ 0 & \bar{A}_{m-r} - s_0 I_{m-r} \end{bmatrix} \begin{bmatrix} \bar{u}^{(2)} \\ \tilde{u}^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & \hat{G}_{12} \\ -\hat{G}_{12}^T & \hat{G}_{22} \end{bmatrix} \begin{bmatrix} \bar{u}^{(1)} \\ \tilde{u}^{(1)} \end{bmatrix} - \underbrace{\begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} \\ \hat{R}_{12}^T & \hat{R}_{22} \end{bmatrix}}_{\tilde{T}^T R \tilde{T}} \begin{bmatrix} \bar{u}^{(0)} \\ 0 \end{bmatrix} + \lambda^{(4)} \begin{bmatrix} \bar{u}^{(0)} \\ 0 \end{bmatrix}, \tag{2.36}$$

where we have denoted

$$\hat{R}_{11} = \hat{R}_{11}^T = \tilde{T}_r R \tilde{T}_r, \quad \hat{R}_{12} = \tilde{T}_r R \tilde{T}_{m-r}, \quad \text{and} \quad \hat{R}_{22} = \tilde{T}_{m-r} R \tilde{T}_{m-r}. \tag{2.37}$$

Taking the first r equations in (2.36) and using (2.32), we get

$$(\hat{G}_{12} \hat{D} \hat{G}_{12}^T + \hat{R}_{11} - \lambda^{(4)} I_r)\bar{u}^{(0)} = 0, \tag{2.38}$$

and taking the last $(m - r)$ equations in (2.36) and again using (2.32) we get

$$\tilde{u}^{(2)} = -\hat{D} \hat{G}_{12}^T \bar{u}^{(1)} - \hat{D}(\hat{G}_{22} \hat{D} \hat{G}_{12}^T + \hat{R}_{12}^T)\bar{u}^{(0)}. \tag{2.39}$$

From (2.38), the r eigenvalues of the r by r symmetric matrix $S_1 = \hat{G}_{12} \hat{D} \hat{G}_{12}^T + \hat{R}_{11}$ are the coefficients $\lambda^{(4)}$ in expansion (2.3) with $\lambda^{(1)} = 0$, $\lambda^{(2)} = s_0$ and $\lambda^{(3)} = 0$. Then

$$\mu_i(\varepsilon) = \lambda_0 + \varepsilon^2 s_0 + \varepsilon^4 \lambda_i^{(4)} + o(\varepsilon^4), \quad i = 1, \ldots, r, \tag{2.40}$$

are the eigenvalues of the matrix $K + \varepsilon N$. Thus, in view of lemma 2.4, we have the next assertion.

Result 2.11. *If $\hat{N}_{11} = 0$, $\hat{G}_{11} = 0$, and all eigenvalues of the r by r symmetric matrix*

$$S_1 = S_1^T = \hat{G}_{12} \hat{D} \hat{G}_{12}^T + \hat{R}_{11}, \tag{2.41}$$

where $\hat{G}_{12}$, $\hat{D}$ and $\hat{R}_{11}$ are determined by (2.29), (2.32), (2.35) and (2.37), respectively, are distinct, then the system described by equation (1.4) is stable for arbitrarily small values of $|\varepsilon|$.

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In the following step, we assume that the symmetric matrix S_1 has a single eigenvalue s_{10} of multiplicity p. In this case, (2.38) becomes

$$(S_1 - s_{10} I_r)\bar{u}^{(0)} = 0, \tag{2.42}$$

while from (2.21) for $r = 5$, we get setting $\lambda^{(1)} = \lambda^{(3)} = 0$, $\lambda^{(2)} = s_0$ and $\lambda^{(4)} = s_{10}$

$$(S - s_0 I_m)\bar{w}^{(3)} = \hat{N}_{12} D \hat{N}_{22} \tilde{w}^{(3)} - s_0 \hat{N}_{12} D \tilde{w}^{(2)} + s_{10} \bar{w}^{(1)} + \lambda^{(5)} \bar{w}^{(0)}. \tag{2.43}$$

Using (2.22), we get $\tilde{w}^{(3)} = D\hat{N}_{12}^T \bar{w}^{(2)} - D\hat{N}_{22}\tilde{w}^{(2)} + s_0 D^2 \hat{N}_{12}^T \bar{w}^{(0)}$, while $\tilde{w}^{(j)}, j = 1, 2$, are given in (2.13) and (2.18). In view of these expressions, (2.43) becomes

$$\begin{aligned}(S - s_0 I_m)\bar{w}^{(3)} &= \underbrace{\hat{N}_{12} D \hat{N}_{22} D N_{12}^T}_{G} \bar{w}^{(2)} - \underbrace{[s_0 \hat{N}_{12} D^2 \hat{N}_{12}^T + \hat{N}_{12} (D\hat{N}_{22})^2 D \hat{N}_{12}^T]}_{R} \bar{w}^{(1)} \\ &\quad + \underbrace{[s_0 \hat{N}_{12} D (D\hat{N}_{22} D + \hat{N}_{22} D^2) \hat{N}_{12}^T + \hat{N}_{12}(D\hat{N}_{22})^3 D\hat{N}_{12}^T]}_{Q} \bar{w}^{(0)} + s_{10}\bar{w}^{(1)} + \lambda^{(5)} \bar{w}^{(0)} \\ &= G\bar{w}^{(2)} - R\bar{w}^{(1)} + Q\bar{w}^{(0)} + s_{10}\bar{w}^{(1)} + \lambda^{(5)}\bar{w}^{(0)},\end{aligned} \tag{2.44}$$

in which the m by m skew-symmetric matrix

$$Q = -Q^T = s_0 \hat{N}_{12} D (D\hat{N}_{22} + \hat{N}_{22} D) D \hat{N}_{12}^T + \hat{N}_{12} (D\hat{N}_{22})^3 D \hat{N}_{12}^T. \tag{2.45}$$

Once again writing $\bar{w}^{(j)} = \tilde{T}[\bar{u}^{(j)T} \quad \tilde{u}^{(j)T}]^T, j = 1, 2, 3$, premultiplying (2.44) by $\tilde{T}^T$, and noting from (2.32) that $\tilde{u}^{(0)} = 0$, we get

$$\begin{aligned}&\begin{bmatrix} 0 & 0 \\ 0 & \bar{\Lambda}_{m-r} - s_0 I_{m-r} \end{bmatrix} \begin{bmatrix} \bar{u}^{(3)} \\ \tilde{u}^{(3)} \end{bmatrix} = \begin{bmatrix} 0 & \hat{G}_{12} \\ -\hat{G}_{12}^T & \hat{G}_{22} \end{bmatrix} \begin{bmatrix} \bar{u}^{(2)} \\ \tilde{u}^{(2)} \end{bmatrix} - \begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} \\ \hat{R}_{12}^T & \hat{R}_{22} \end{bmatrix} \begin{bmatrix} \bar{u}^{(1)} \\ \tilde{u}^{(1)} \end{bmatrix} + \underbrace{\begin{bmatrix} \hat{Q}_{11} & \hat{Q}_{12} \\ \hat{Q}_{12}^T & \hat{Q}_{22} \end{bmatrix}}_{\tilde{T}^T Q \tilde{T}} \begin{bmatrix} \bar{u}^{(0)} \\ 0 \end{bmatrix} \\ &\qquad + s_{10} \begin{bmatrix} \bar{u}^{(1)} \\ \tilde{u}^{(1)} \end{bmatrix} + \lambda^{(5)} \begin{bmatrix} \bar{u}^{(0)} \\ 0 \end{bmatrix} = 0.\end{aligned}$$

Taking the first r equations, we get

$$\hat{G}_{12}\tilde{u}^{(2)} - \hat{R}_{11}\bar{u}^{(1)} - \hat{R}_{12}\tilde{u}^{(1)} + \hat{Q}_{11}\bar{u}^{(0)} + s_{10}\bar{u}^{(1)} + \lambda^{(5)}\bar{u}^{(0)} = 0,$$

where $\hat{Q}_{11} = \tilde{T}_r^T Q \tilde{T}_r$. Upon using (2.32) and (2.39) for $\tilde{u}^{(1)}$ and $\tilde{u}^{(2)}$, we get

$$(\underbrace{\hat{G}_{12}\hat{D}\hat{G}_{12}^T + \hat{R}_{11}}_{S_1} - s_{10} I_r)\bar{u}^{(1)} = [\underbrace{\hat{Q}_{11} - \hat{G}_{12}\hat{D}(\hat{G}_{22}\hat{D}\hat{G}_{12}^T + \hat{R}_{12}^T) + \hat{R}_{12}\hat{D}\hat{G}_{12}^T}_{\Omega} + \lambda^{(5)} I_r]\bar{u}^{(0)},$$

or,

$$(S_1 - s_{10} I_r)\bar{u}^{(1)} = [\Omega + \lambda^{(5)} I_r]\bar{u}^{(0)}, \tag{2.46}$$

where

$$\Omega = -\Omega^T = \hat{Q}_{11} - \hat{G}_{12}\hat{D}(\hat{G}_{22}\hat{D}\hat{G}_{12}^T + \hat{R}_{12}^T) + \hat{R}_{12}\hat{D}\hat{G}_{12}^T, \quad \text{and} \quad \hat{Q}_{11} = \tilde{T}_r^T Q \tilde{T}_r. \tag{2.47}$$

Note that the 'structure' of the two equations (2.42) and (2.46) is the same as the structure of the two equations (2.26) and (2.27). Thus, by following a procedure similar to that adopted earlier, we will be led ultimately to a result analogous to that given in (2.32). We now follow this procedure that was used earlier.

Let $\tilde{\tilde{T}} = [\tilde{\tilde{T}}_p | \tilde{\tilde{T}}_{r-p}]$ be an r by r orthogonal matrix, where the $r \times p$ submatrix $\tilde{\tilde{T}}_p$ contains p eigenvectors of S_1 corresponding to the multiple eigenvalue s_{10}, and the $r \times (r-p)$ submatrix $\tilde{\tilde{T}}_{r-p}$

contains the remainder $r-p$ of the eigenvectors of S_1. The matrix $\tilde{\tilde{T}}$ transforms matrices (2.41) and (2.47) to the forms

$$\hat{S}_1 = \tilde{\tilde{T}}^T S_1 \tilde{\tilde{T}} = \mathrm{diag}(s_{10} I_p, \bar{\bar{\Lambda}}_{r-p}), \quad \hat{\Omega} = \tilde{\tilde{T}}^T \Omega \tilde{\tilde{T}} = \begin{bmatrix} \hat{\Omega}_{11} & \hat{\Omega}_{12} \\ -\hat{\Omega}_{12}^T & \hat{\Omega}_{22} \end{bmatrix}, \tag{2.48}$$

where the diagonal matrix $\bar{\bar{\Lambda}}_{r-p} = \tilde{\tilde{T}}_{r-p}^T S_1 \tilde{\tilde{T}}_{r-p}$ and $\hat{\Omega}_{11} = \tilde{\tilde{T}}_p^T \Omega \tilde{\tilde{T}}_p$, $\hat{\Omega}_{12} = \tilde{\tilde{T}}_p^T \Omega \tilde{\tilde{T}}_{r-p}$, $\hat{\Omega}_{22} = \tilde{\tilde{T}}_{r-p}^T \Omega \tilde{\tilde{T}}_{r-p}$. Substituting $\bar{u}^{(j)} = \tilde{\tilde{T}}[\bar{v}^{(j)T} \;\; \tilde{v}^{(j)T}]^T$, where $\bar{v}^{(j)}$ and $\tilde{v}^{(j)}$ are p and $(r-p)$ dimensional vectors, respectively, in equations (2.42) and (2.46) and premultiplyng these by $\tilde{\tilde{T}}^T$ we get (analogous to the result in (2.32))

$$\tilde{v}^{(0)} = 0, \tilde{v}^{(1)} = -\hat{\hat{D}} \hat{\Omega}_{12}^T \bar{v}^{(0)}, (\hat{\Omega}_{11} + \lambda^{(5)} I_p) \bar{v}^{(0)} = 0, \tag{2.49}$$

where $\hat{\hat{D}} = (\bar{\bar{\Lambda}}_{r-p} - s_{10} I_{r-p})^{-1}$.

It follows from the last equation in (2.49) that p eigenvalues of the p by p skew-symmetric matrix $\hat{\Omega}_{11} = \tilde{\tilde{T}}_p^T \Omega \tilde{\tilde{T}}_p$ are the coefficients $\lambda^{(5)}$ in expansion (2.3) with $\lambda^{(1)} = 0$, $\lambda^{(2)} = s_0$, $\lambda^{(3)} = 0$ and $\lambda^{(4)} = s_{10}$. If this matrix is non-zero, then it has at least one pair of conjugate purely imaginary eigenvalues $\pm i\omega$ with $\omega > 0$, so that the matrix $K + \varepsilon N$ has at least one complex conjugate pair of eigenvalues of the form

$$\mu(\varepsilon) = \lambda_0 + \varepsilon^2 s_0 + \varepsilon^4 s_{10} \pm i\varepsilon^5 \omega + o(\varepsilon^5). \tag{2.50}$$

This leads to the following result.

Result 2.12. *If $\hat{N}_{11} = 0$, $\hat{G}_{11} = 0$ and $\hat{\Omega}_{11} = \tilde{\tilde{T}}_p^T \Omega \tilde{\tilde{T}}_p \neq 0$ then the system described by equation (1.4) is unstable by flutter for arbitrarily small non-zero values of $|\varepsilon|$.*

If $\hat{\Omega}_{11} = \tilde{\tilde{T}}_p^T \Omega \tilde{\tilde{T}}_p = 0$, the above procedure can be continued in the same manner.

Note that at every step in the above reduction procedure, a matrix of simple structure (more precisely, symmetric or skew-symmetric) appears, which ensures Taylor's expansions of eigenvalues in ε. By contrast, when the perturbation matrix is neither symmetric nor skew-symmetric, fractional powers of ε may appear, as shown in [15].

Remark 2.13. If $\hat{N}_{11} = 0$ and $\hat{N}_{12} = 0$, it is clear that λ_0 remains a semi-simple eigenvalue of multiplicity m of the matrix $\hat{\Lambda} + \varepsilon \hat{N}$ and, consequently, the system described by equation (1.4) is stable for arbitrarily small values of $|\varepsilon|$.

Obviously, in this case, the system in normal coordinates is decoupled into two subsystems, one of which is an m-dimensional stable purely potential system independent of ε and the other is an (n–m)-dimensional circulatory system that is stable for sufficiently small $|\varepsilon|$.

Observe that if $\hat{N}_{11} = 0$, then the first non-zero coefficient $\lambda^{(2j-1)}$ in expression (2.3) is an eigenvalue of some skew-symmetric matrix, but if in addition $\hat{N}_{22} = 0$, it is the zero matrix, while $\lambda^{(2j)}$ are the eigenvalues of some symmetric matrices.

Therefore, we can state the following.

Result 2.14. *If $\hat{N}_{11} = 0$ and $\hat{N}_{22} = 0$, then the system described by equation (1.4) is stable for arbitrarily small values of $|\varepsilon|$.*

Remark 2.15. This result is also valid when all diagonal elements of the matrix Λ_{n-m} are not different.

A direct consequence of the results 2.5 and 2.14 is the following assertion.

Corollary 2.16. *Let $m = n-1$. Then the system described by equation (1.4) is unstable for arbitrarily small non-zero values of $|\varepsilon|$ if and only if $\hat{N}_{11} \neq 0$.*

We note that this result also follows from [10, see Theorem 3].

Remark 2.17. Suppose that $\omega = 0$ in (2.50), i. e. $\hat{\Omega}_{11} = 0$, and that the system has a flutter mode corresponding to λ_0. It is easy to see that the rate of exponential growth of the flutter mode is $\gamma = c|\varepsilon|^7 + o(|\varepsilon|^7)$, $c = const > 0$, and, consequently, in a large time interval (approx. ε^{-6}), we have a small increase ($\sim\varepsilon$) in the amplitude of the oscillations. Therefore, from a practical (engineering) point of view, it can be assumed that the above results close the problem posed in §2.

Example 2.18. Let

$$K = \mathrm{diag}(I_2, 3, 2, 9), \quad N = \begin{bmatrix} \nu J_2 & N_{12} \\ -N_{12}^T & N_{22} \end{bmatrix}, \tag{2.51}$$

where $\nu \in \Re$ and

$$J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad N_{12} = \begin{bmatrix} 2 & 0 & -4 \\ 0 & 2 & 0 \end{bmatrix}, \quad N_{22} = \begin{bmatrix} 0 & -\sqrt{11}/2 & 0 \\ \sqrt{11}/2 & 0 & \alpha \\ 0 & -\alpha & 0 \end{bmatrix}, \quad \alpha \in \Re. \tag{2.52}$$

For this example $n = 5$, $\lambda_0 = 1$ and $m = 2$. If $\nu \neq 0$, then according to result 2.5, the system (1.4), (2.51), (2.52) is unstable by flutter. Let now $\nu = 0$. Then, it is easy to see that $D = \mathrm{diag}(2, 1, 8)$ and the matrix (2.19) is $S = 4I_2$, i.e. the matrix S has the double eigenvalue $s_0 = 4$. The matrix (2.28) becomes $G = (\alpha - \sqrt{11})J_2$. Consequently, if $\alpha \neq \sqrt{11}$, then in view of result 2.10, instability follows. Finally, if $\alpha = \sqrt{11}$, we get that the symmetric matrix (2.41) becomes $S_1 = 5I_2$, and it clearly has the double eigenvalue $s_{10} = 5$. Further, the matrix (2.47) becomes $\Omega = -3/2\sqrt{11}J_2$, and according result 2.12 again instability follows.

Thus, if $|\varepsilon|$ is sufficiently small, then the system (1.4), (2.51)–(2.52) is unstable by flutter for every value of the real parameters α and ν.

3. Systems with a small number of degrees of freedom

In this section for $n < 5$, based on the results in §2, we will describe all skew-symmetric matrices N, such that circulatory forces determined by εN, where the parameter ε is arbitrarily small, cause flutter instability(stability) in potential systems with multiple eigenvalues. By flutter instability (stability), we mean here instability (stability) caused by arbitrarily small non-zero values of the parameter $|\varepsilon|$. For simplicity, we assume that the system (1.4) are described in normal coordinates so that the potential matrices are diagonal ($K = \hat{\Lambda}$) and $N = \hat{N}$. Obviously, if $m = n$, then, according to Merkin's theorem, the flutter instability follows for every non-zero N. Thus, in what follows we suppose that $m < n$.

(a) $n = 3, m = 2$

In this case,

$$\hat{\Lambda} = \mathrm{diag}(\lambda_0 I_2, \lambda_3), \quad \hat{N} = \begin{bmatrix} \nu J_2 & \hat{N}_{12} \\ -\hat{N}_{12}^T & 0 \end{bmatrix}, \tag{3.1}$$

where

$$J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and $0 < \lambda_0 \neq \lambda_3 \in \Re$, $\nu \in \Re$ and $\hat{N}_{12}$ is a real 2×1 matrix. It follows from result 2.5 and corollary 2.16 that the systems (1.4) and (3.1) is unstable by flutter if and only if $\nu \neq 0$.

(b) $n = 4$

(i) $m = 3$

In this case,

$$\hat{\Lambda} = \mathrm{diag}(\lambda_0 I_3, \lambda_4), \quad \hat{N} = \begin{bmatrix} \hat{N}_{11} & \hat{N}_{12} \\ -\hat{N}_{12}^T & 0 \end{bmatrix}, \tag{3.2}$$

where $0 < \lambda_0 \neq \lambda_4 \in \Re$, $\hat{N}_{11}$ is 3×3 skew-symmetric and $\hat{N}_{12}$ is 3×1 matrix. According to result 2.5 and corollary 2.16, the system (1.4), (3.2) is unstable by flutter if and only if $\hat{N}_{11} \neq 0$.

(ii) $m = 2$

(a) The case of a single multiple eigenvalue

In this case,

$$\hat{\Lambda} = \mathrm{diag}(\lambda_0 I_2, \lambda_3, \lambda_4), \quad \hat{N} = \begin{bmatrix} \nu_1 J_2 & \hat{N}_{12} \\ -\hat{N}_{12}^T & \nu_2 J_2 \end{bmatrix}, \tag{3.3}$$

where $0 < \lambda_0 \neq \lambda_3 \neq \lambda_4 \in \Re, \nu_1, \nu_2 \in \Re$ and $\hat{N}_{12}$ is a real 2×2 matrix.
If $\nu_1 \neq 0$, then, in view of result 2.5, the system (1.4), (3.3) is unstable by flutter.
Let $\nu_1 = 0$. Then, it is easy to show that the matrix $S = \hat{N}_{12} D \hat{N}_{12}^T$, where $D = \mathrm{diag}[1/(\lambda_3 - \lambda_0), 1/(\lambda_4 - \lambda_0)]$, has a double eigenvalue if and only if one of the two following conditions holds:

$$(1) \quad (\lambda_3 - \lambda_0)(\lambda_4 - \lambda_0) > 0, \quad \hat{N}_{12} = \begin{bmatrix} a & \mp\eta b \\ b & \pm\eta a \end{bmatrix} \tag{3.4}$$

and

$$(2) \quad (\lambda_3 - \lambda_0)(\lambda_4 - \lambda_0) < 0, \quad \hat{N}_{12} = \begin{bmatrix} a & \pm\eta a \\ b & \pm\eta b \end{bmatrix}. \tag{3.5}$$

Here $\eta = \sqrt{|\lambda_4 - \lambda_0/\lambda_3 - \lambda_0|}$ and $a, b \in \Re$. If both conditions (3.4) and (3.5) fail, then the symmetric matrix S has distinct eigenvalues, and, according to result 2.8, the system (1.4), (3.3) is stable. Under conditions (3.4), the matrix S has the double eigenvalue $s_0 = (a^2 + b^2)/(\lambda_3 - \lambda_0)$, and the skew-symmetric matrix G determined by (2.28) is

$$G = \pm \frac{\nu_2 \eta (a^2 + b^2)}{(\lambda_3 - \lambda_0)(\lambda_4 - \lambda_0)} J_2.$$

From this and in view of result 2.10, it follows that the system is unstable by flutter if $\nu_2(a^2 + b^2) \neq 0$. However, if $\nu_2(a^2 + b^2) = 0$, then according to result 2.14 and remark 2.13, the system is stable.

On the other hand, under conditions (3.5) we have $S = 0, G = 0, R = 0$ and so on, which indicate that λ_0 remains a double semi-simple eigenvalue. Indeed, it is easy to confirm that in this case the matrix $(\hat{\Lambda} + \varepsilon \hat{N})$ has double eigenvalue λ_0 with corresponding eigenvectors of the forms

$$w^{(j)} = \begin{bmatrix} u^{(j)} \\ \varepsilon (D + \varepsilon \nu_2 J_2)^{-1} \hat{N}_{12}^T u^{(j)} \end{bmatrix}, \quad j = 1, 2,$$

where $u^{(1)}$ and $u^{(2)}$ are linearly independent two-dimensional real vectors. Consequently, under conditions (3.5), the system (3.3) is stable.

Thus, the system (1.4), (3.3) is unstable by flutter if and only if either $\nu_1 \neq 0$ or the conditions (3.4) hold and $\nu_2(a^2 + b^2) \neq 0$.

(b) The case of two multiple eigenvalues

In this case,

$$\hat{\Lambda} = \mathrm{diag}(\lambda_1 I_2, \lambda_2 I_2), \quad \hat{N} = \begin{bmatrix} \nu_1 J_2 & \hat{N}_{12} \\ -\hat{N}_{12}^T & \nu_2 J_2 \end{bmatrix}, \tag{3.6}$$

where $0 < \lambda_1 \neq \lambda_2 \in \Re$, $\nu_1, \nu_2 \in \Re$ and $\hat{N}_{12}$ is a real 2×2 matrix. It follows from (3.6), in view of remark 2.7, that system (1.4), (3.6) is unstable by flutter if either $\nu_1 \neq 0$ or $\nu_2 \neq 0$. If $\nu_1 = \nu_2 = 0$, then according to result 2.14, the system (1.4), (3.6) is stable. Note that this is in accordance with a result of [10, Theorem 3].

4. Conclusion

This paper addresses the question of the stability of a stable potential system to infinitesimal circulatory perturbations. A stable potential system that has one multiple eigenvalue (with multiplicity greater than 1) is considered, the other eigenvalues being all distinct. Such a situation is not uncommon in large-scale engineered multi-degree-of-freedom (MDOF) systems such as ships, spacecraft, aircraft and building structures. The results obtained herein are therefore important from a practical standpoint since they can play an important role in our understanding of complex physical phenomena and in the development of safer engineering designs.

Using a perturbation expansion, the bifurcations of the multiple eigenvalues under vanishingly small circulatory perturbations are investigated. While it is often believed that all such circulatory perturbations make the system unstable, it is shown that this may not necessarily be true. This is because most perturbation studies to date that have dealt with circulatory perturbations deal with linear perturbations, and the perturbation expansion stops at the linear term in the so-called small perturbation parameter, ε.

The detailed investigation carried out here on general MDOF systems in which the perturbation expansion is continued well beyond the linear perturbation regime shows that the question of stability is much subtler than was previously envisioned, and a fairly complex stability picture of potential systems under infinitesimal circulatory perturbations emerges. The explicit results provided show a somewhat aesthetic alternation in the 'structure' of stability and instability which is dependent on the nature of the circulatory perturbatory matrix and its interaction with all the frequencies of vibration of the potential system. It is to emphasize this alternating character of the structure of the stability (and instability) that the words stability and instability are both included in the title of the paper.

While the paper principally focuses on MDOF systems that may be modelled by hundreds or thousands of degrees of freedom, the more detailed approach developed here also yields new insights into their lower dimensional counterparts, and a near-complete stability analysis for such systems is consequently obtained. While such low-dimensional systems are rare in the description of natural and engineered systems—unless there are significant symmetries and/or constraints on the system—reduced-order models are often used in obtaining a preliminary understanding of physical phenomena and in the initial design phases of large-scale engineered MDOF systems.

Compared with earlier perturbation studies, the treatment undertaken herein is much more comprehensive and provides several new results that appear to bring us closer to a thorough understanding of the stability of potential systems to infinitesimal circulatory positional perturbations, an area of considerable interest to physicists, mathematician and engineers. In order to illustrate the current state of our knowledge and its improvement through the theory developed herein, several examples are provided.

Data accessibility. This article has no additional data.

Authors' contributions. R.B. and F.U. jointly worked on §1 of the paper. R.B. sketched out the initial framework of the central problem to be solved given in §1. F.U. and R.B. jointly worked on the proofs for all the results obtained in §2. F.U. finalized these proofs. R.B. worked on §3 and F.U. rederived and checked the computations. R.B. and F.U. jointly worked on §4 of the paper. Both authors gave final approval for publication and agree to be held accountable for the work performed therein.

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